UNIVERSITY ENTRANCE EXAMINATION 2019

MATHEMATICS ('A' LEVEL EQUIVALENT)

Time allowed : 2 hours

INSTRUCTIONS TO CANDIDATES

1. This examination paper has TWO (2) sections – A and B, and comprises **FIFTEEN (15)** printed pages.
2. Attempt all sections.
3. Answer all questions in Section A. Indicate your answers on the answer paper provided. Each question carries 2 marks. Marks will not be deducted for wrong answers.
4. Answer FOUR (4) questions from Section B with not more than THREE (3) from any one option. Write your answers on the answer paper provided. Begin each question on a fresh sheet of paper. Write the question number clearly. Each question carries 15 marks.
5. A non-programmable scientific calculator may be used. However, candidates should lay out systematically the various steps in the calculation.
6. At the end of the examination, attach the cover paper on top of your answer script. Complete the information required on the cover page and tie the papers together with the string provided. The colour of the cover paper for this examination is **GREEN**.
7. Do not take any paper, including the question paper and unused answer paper, out of the examination hall.

SECTION A (40 Marks)

Answer all questions in this section. Each question carries 2 marks.

1. The points P, Q and R have coordinates $(2, 7)$, (a, b) and $(5, 16)$ respectively. Suppose P, Q and R are collinear. Then the value of $\dfrac{a-2}{7-b}$ is

   (A) $-\frac{1}{4}$

   (B) $-\frac{1}{3}$

   (C) $\frac{1}{4}$

   (D) $\frac{1}{3}$

   (E) none of the above

2. The expressions $x^4 - 2x^3 - Ax^2 - Bx - 8$ and $Ax^4 - 5x + B$ have a common factor $x - 2$. The values of A and B are

   (A) $A = 1$ and $B = 6$

   (B) $A = 1$ and $B = -6$

   (C) $A = -1$ and $B = 6$

   (D) $A = -1$ and $B = -6$

   (E) none of the above

3. The range of values of p for which $8 - 3x - x^2 \leq p$ for all real values of x is

   (A) $p \geq -\frac{23}{4}$

   (B) $p \leq -\frac{23}{4}$

   (C) $p \geq \frac{41}{4}$

   (D) $p \leq \frac{41}{4}$

   (E) none of the above

4. Suppose $2x^3 + 3x^2 - 14x - 5 = (Px + Q)(x + 3)(x + 1) + R$ for all values of x. Then

(A) $P = 2$, $Q = -5$ and $R = -20$

(B) $P = 2$, $Q = -5$ and $R = 10$

(C) $P = 2$, $Q = 5$ and $R = -20$

(D) $P = 2$, $Q = 5$ and $R = 10$

(E) none of the above

5. The derivative of

$$(x + 3)\sqrt{2x - 3}$$

with respect to x is

(A) $\dfrac{3x}{\sqrt{2x - 3}}$

(B) $\dfrac{3x}{2\sqrt{2x - 3}}$

(C) $\dfrac{3x + 1}{\sqrt{2x - 3}}$

(D) $\dfrac{3x + 1}{2\sqrt{2x - 3}}$

(E) none of the above

6. The inequality $64 - |7x| \geq |9x|$ has solution

(A) $-2 \leq x \leq 2$

(B) $-4 \leq x \leq 4$

(C) $0 \leq x \leq 4$

(D) $-8 \leq x \leq 8$

(E) none of the above

7. A tennis team of 4 men and 4 women is to be selected from 6 men and 7 women. It was decided that two particular women A and B must either be selected together or not selected at all. The number of ways in which this can be done is

(A) 125

(B) 225

(C) 375

(D) 525

(E) none of the above

8. Suppose $-9 \leq x \leq 7$ and $-6 \leq y \leq 8$. Then the smallest value of $y^2 - (3-x)^2$ is

(A) -208

(B) -145

(C) -144

(D) -80

(E) none of the above

9. The curve for which $\dfrac{dy}{dx} = 4x + k$, where k is a constant, has a turning point at $(-2, -1)$. The value of k is

(A) 8

(B) $\dfrac{1}{4}$

(C) $-\dfrac{1}{4}$

(D) -8

(E) none of the above

10. The function $f(x) = \frac{2}{3}ax + b$ is such that $f(\frac{1}{2}) = 4$ and $f^{-1}(5) = \frac{3}{2}$. The value of a is

(A) $\frac{1}{3}$

(B) $\frac{1}{2}$

(C) $\frac{2}{3}$

(D) $\frac{3}{2}$

(E) none of the above

11. The vector $\begin{pmatrix} 2w \\ 3 \end{pmatrix}$ is perpendicular to the vector $\begin{pmatrix} -7 \\ 3v \end{pmatrix}$. The value of $\dfrac{13v + 14w}{15v + 14w}$ is

(A) $-\frac{11}{13}$

(B) $-\frac{11}{12}$

(C) $\frac{11}{12}$

(D) $\frac{11}{13}$

(E) none of the above

12. Two fair dice are thrown and the score on each die is noted. The probability that the product of the scores is a multiple of 3 is

(A) $\frac{1}{9}$

(B) $\frac{5}{36}$

(C) $\frac{5}{9}$

(D) $\frac{2}{3}$

(E) none of the above

13. The values of w for which the equation $(w+1)x^2+4wx+9=0$ has equal roots are

(A) $-\frac{3}{4}$ and -3

(B) $-\frac{3}{4}$ and 3

(C) $\frac{3}{4}$ and -3

(D) $\frac{3}{4}$ and 3

(E) none of the above

14. The minimum value of the function $f(x) = (5\cos x - 7)^2 - 2$ is

(A) -2

(B) 2

(C) 13

(D) 23

(E) none of the above

15. The number of ways in which the letters of the word *INCLUDE* can be arranged so that all the consonants are together is

(A) 72

(B) 144

(C) 576

(D) 720

(E) none of the above

16. Which option corresponds to the partial fraction decomposition of the rational function $\dfrac{-17}{12x^2 - x - 6}$?

(A) $-\dfrac{4}{4x-3} - \dfrac{3}{3x+2}$

(B) $\dfrac{4}{4x-3} - \dfrac{3}{3x+2}$

(C) $-\dfrac{4}{4x-3} + \dfrac{3}{3x+2}$

(D) $\dfrac{4}{4x-3} + \dfrac{3}{3x+2}$

(E) none of the above

17. Which of the following is the result of completing the square of the expression $-3x^2 + 8x - 5$?

(A) $-3\left(x - \dfrac{4}{3}\right)^2 - \dfrac{31}{3}$

(B) $-3\left(x - \dfrac{4}{3}\right)^2 - \dfrac{16}{3}$

(C) $-3\left(x - \dfrac{4}{3}\right)^2 - \dfrac{1}{3}$

(D) $-3\left(x - \dfrac{4}{3}\right)^2 + \dfrac{1}{3}$

(E) none of the above

18. The equation of a curve is $y = 2x + \frac{6}{x}$. The equation of the normal to the curve at the point $x = 2$ is

(A) $y = -2x - 11$

(B) $y = -2x + 11$

(C) $y = 2x - 11$

(D) $y = 2x + 11$

(E) none of the above

19. The first three terms of an arithmetic progression are $\lg(3x)$, $\lg(3x+2)$ and $\lg(3x+16)$. The value of x is

(A) $\frac{1}{9}$

(B) $\frac{1}{3}$

(C) 1

(D) 3

(E) none of the above

20. The line $y = x - 6$ meet the curve $y^2 = 8x$ at the points A and B. The length of AB is

(A) $8\sqrt{2}$

(B) $12\sqrt{2}$

(C) $16\sqrt{2}$

(D) $24\sqrt{2}$

(E) none of the above

SECTION B (60 Marks)

Answer FOUR (4) questions with not more than THREE (3) from any one option.

Option (a) - Pure Mathematics

21(a). Find the integral $\int 2x \ln(x+1)\ dx$. [5 Marks]

21(b). Find the integral $\int \frac{\cos 2\theta}{\sin^2 \theta}\ d\theta$. [5 Marks]

21(c). Find $\int_0^{\frac{\pi}{4}} \frac{\cos x + 6 \sin x}{3 \cos^3 x}\ dx$. [5 Marks]

22(a). Solve the following inequalities

(i) $|6x-1| > |3x-4|$, [4 Marks]

(ii) $\frac{3}{|2x|+1} < |x|$. [6 Marks]

22(b). Prove by induction that

$$\sum_{r=1}^{n} (r!)(r^2+1) = n\big((n+1)!\big).$$

[5 Marks]

23(a). By using the substitution $v = x - y$, or otherwise, find the general solution of the differential equation

$$\frac{dy}{dx} + \left(1 + (y - x)^2\right) \sin x = 1,$$

expressing y in terms of x. [7 Marks]

23(b). Solve the differential equation

$$\frac{dy}{dx} = (4 + y^2) \cos^2 x,$$

given that $y = 0$ when $x = 0$. [8 Marks]

24. Relative to the origin O, points P, Q and R have position vectors $\mathbf{i} + \mathbf{j}$, $\mathbf{i} + \mathbf{k}$ and $\mathbf{j} + \mathbf{k}$ respectively. The origin and the points P, Q and R form a regular tetrahedron. The line L has equation $\mathbf{r} = t\mathbf{i} + t\mathbf{j} + t\mathbf{k}$, where t is a real parameter.

(i) Find a vector equation of the line joining O to M, where M is the mid-point of PQ. [3 Marks]

(ii) Let N be the foot of the perpendicular from R to the line OM. Find the position vector of N. Hence, or otherwise, find a vector equation of the line joining R and N. [6 Marks]

(iii) Let C be the point of intersection of the lines L and RN. Find the position vector of C. [3 Marks]

(iv) Find the angle OCR giving your answer correct to the nearest $0.1°$. [3 Marks]

Option (b) - Particle Mechanics

[In this section, take the acceleration due to gravity to be 9.8 ms^{-2}, unless otherwise stated. Give non-exact numerical answers correct to three significant figures, unless otherwise specified.]

25 (a). Two particles A and B are moving in the same direction in the same straight line on a smooth horizontal plane. The particle A has mass 2 kg and the particle B has mass 3 kg. Particle A moving with speed $4\,\text{ms}^{-1}$ catches up with particle B whose speed is $v\,\text{ms}^{-1}$. Immediately after the collision, B has speed $\frac{16}{5}\,\text{ms}^{-1}$. Given that the impulse acting on A has magnitude $\frac{18}{5}$ Ns, find v and the loss of kinetic energy in the collision. [7 Marks]

25 (b). Two fixed points A and B are in the same vertical line with A at a distance $2L$ metres above B. A small smooth ring P of mass m kg is threaded on a light elastic string of modulus of elasticity $3mg$ N and natural length L metres, and both ends of the string are fixed at A. The ring P is also attached to one end of another inextensible string of length L metres with the other end fixed at B. Find the tension in the string PB when the system is in equilibrium with A, P and B in the same vertical line. [4 Marks]

The inextensible string is being cut while the system is in equilibrium. Find in terms of g and L the speed of P just before it hits A. [4 Marks]

26. [Take the acceleration due to gravity to be $10\,\text{ms}^{-2}$ in this question.]

A stone projected with speed $u\,\text{ms}^{-1}$ and angle of elevation α from a point O on horizontal ground moves freely under gravity. It just clears the top of a vertical wall of height 9 metres at a horizontal distance of 30 metres from O. It hits the ground at a horizontal distance of 12 metres from the wall.

Calculate $u, \tan\alpha$ and the greatest height reached by the stone. Find also the magnitude and direction of velocity of the stone when it is at the top of the wall and the time of flight of the stone. [15 Marks]

27. The engine of a car of mass m kg can work at a maximum power of P watts. The car begins accelerating from rest with maximum power P and moves on a horizontal road.

   (a) The magnitude of the resistance to motion of the car is kv^2 N, where k is a positive constant and $v\,\mathrm{ms}^{-1}$ is the speed of the car. Express the maximum speed of the car in terms of P and k. Find the distance, in metres, travelled by the car from the instant it started from rest to the instant it moves with half its maximum speed in terms of m and k. [8 Marks]

   (b) The magnitude of the resistance to motion of the car is now changed to kv N. Find the new maximum speed that can be attained by the car in terms of P and k. Hence, find the time taken for the car to accelerate from rest to half its maximum speed if it operates at maximum power. [7 Marks]

28. A rigid pole OA of length $4L$ metres has one end O fixed to a horizontal table and the other end A vertically above O. The ends of a light inextensible string of length $4L$ metres are fixed to A and a point B which is at a distance $2L$ metres below A on the pole. A small particle P of mass m kg is fastened to the mid-point of the string and made to rotate with both parts of the string taut in a horizontal circle with angular speed ω radians per second.

   Find the tensions in both parts of the string in terms of m, L, ω and g, where g is the acceleration due to gravity, [5 Marks]

   At a given instant of rotation, both parts of the string are cut. Assuming that there is no air resistance and that $\omega = \sqrt{\frac{g}{3L}}$, find the time which elapses before the particle strikes the table. Suppose the particle strikes the table at the point C. Find the length of OC. [6 Marks]

   Find also the magnitude and direction of the velocity of the particle just before it strikes the table. [4 Marks]

Option (c) - Probability and Statistics

[In this section, probabilities should be expressed as either fractions in lowest terms or decimals with three significant figures.]

29. A fair cubical die has two yellow faces and four blue faces. In one game, the die is allowed to roll repeatedly until a yellow face appears uppermost, subject to a maximum of 4 throws. The random variable B represents the number of times a blue face appears uppermost and the random variable R represents the number of times the die is rolled.

 (i) Find $P(B = 3)$. [2 Marks]

 (ii) Find the probability distribution of B. [3 Marks]

 (iii) Find $E(B)$. [2 Marks]

 (iv) Find $P(R = 4)$. [2 Marks]

 (v) Find $P(R = B)$. [2 Marks]

If the game is repeated 200 times, write down (but do not evaluate) an expression for the probability of obtaining $B = 3$ exactly 25 times. Using an appropriate approximation to estimate this probability. [4 Marks]

30. The continuous random variable X has probability density function given by

$$f(x) = \begin{cases} (1+x)^2, & -1 \le x \le 0, \\ 1 - \frac{3}{4}x, & 0 \le x \le \frac{4}{3}, \\ 0, & \text{otherwise.} \end{cases}$$

(i) Find the cumulative distribution function F of X. Hence, or otherwise, find the value of $P\left(X < 1 \middle| X > -\dfrac{1}{3}\right)$. [7 Marks]

(ii) Find $E(X)$. [3 Marks]

(iii) Forty independent observations of X are taken and M denotes the arithmetic mean of these observations. Given that $Var(X) = 0.136$, state the approximate distribution of M and use it to find the probability that M lies between 0.15 and 0.25. [5 Marks]

31. Tasty Bakery produces muffins in batches of 120. The mean number of muffins with at least 3 raisins in a batch is 61.2. A quality inspector wants to know the probability that, in a randomly selected batch, there are at least 55 but not more than 60 muffins with at least 3 raisins. The number of raisins in any muffin may be assumed to be independent of the number of raisins in any other muffin.

(i) State fully, without evaluating, the distribution that should be used to obtain the exact value of this probability. [3 Marks]

(ii) Using a suitable approximation, find this probability. [5 Marks]

The machine producing the muffins is adjusted so that the mean number of raisins in a muffin is 2.7. Assuming that the raisins occur randomly in the muffins, find the probability that a randomly selected muffin has at least 3 raisins. [3 Marks]

Given that a randomly selected muffin has at most 4 raisins, find the probability that the muffin has at least 3 raisins. [4 Marks]

32. The Mathematics marks of students in Excel High School are normally distributed with mean 58 and standard deviation 18.

(i) Find the probability that a randomly chosen student from Excel High School score less than 45 marks in mathematics. [2 Marks]

(ii) The probability of a randomly chosen student from Excel High School scoring less than k marks is not more than 0.1. Find the greatest value of k giving your answer correct to one decimal place. [4 Marks]

(iii) Two hundreds students are chosen at random from Excel High School. Find the probability that at least forty of them score less than 45 marks. [4 Marks]

The Mathematics marks of students in Noble High School are normally distributed with mean 62 and standard deviation 10. Assume that the Mathematics marks of students in Noble High School are independent of the Mathematics marks of students in Excel High School.

(iv) Find the probability that the total Mathematics marks of two students from Noble High School is more than three times the Mathematics mark of one student from Excel High School. [5 Marks]

END OF PAPER